Int. J. Heat Mass Transfer. Vol. 36, No. 1, pp. 49–56, 1993
Printed in Great Britain

0017-9310/93 \$5.00+0.00

How to distribute a finite amount of insulation on a wall with nonuniform temperature

ADRIAN BEJAN

Department of Mechanical Engineering and Materials Science, Duke University, Durham, NC 27706, U.S.A.

(*Received* 22 *January* 1992 *and in final form* 9 *March* 1992)

Abstract—This paper shows that it is possible to distribute a finite amount of insulation in an optimal way that minimizes the overall heat transfer rate from a nonisothermal wall to the ambient. The optimal insulation thickness for a plane wall varies as the square root of the local wall–ambient temperature difference. Corresponding variational-calculus results are developed for cylindrical walls covered with insulation. The heat loss reduction associated with using the optimal thickness is greater when the wall is plane, as opposed to cylindrical, and when the wall temperature variation in the x direction has a greater second derivative, d^2T/dx^2. It is shown finally that the best insulation for a single-phase stream suspended in an environment of different temperature is the insulation with uniform thickness.

1. INTRODUCTION

IN THIS paper I consider the fundamental question of how a finite amount of insulation material can best be distributed over a wall with nonuniform temperature, in order to minimize the total heat loss from the wall to the ambient. This question is important and interesting. It is important because many industrial applications in which energy conservation is a major concern require insulation for walls that are not isothermal. For example, this is true of the outer wall of a long reheating oven, in which the material that is being heated (e.g. steel laminates) rides slowly on a conveyor belt through the oven, the outer walls of virtually all heat exchangers, storage tanks with thermally stratified liquids (e.g. solar thermal applications), and the lateral surfaces of mechanical supports connecting regions with different temperatures.

Next to the task of conserving energy, the idea that the supply of insulation material is finite is always on the mind of the designer. The purchase, installation and maintenance of an insulation can be expensive. In some cases even the size (weight, volume) of the used insulation material cannot exceed a certain limit. Examples of this kind are airborne applications, and installations where the integrity of the mechanical supports is threatened by the weight of the insulated system (e.g. the suspended insulated duct of Fig. 4).

The idea examined in this paper is interesting as well. I found this only after I started to work on it, because originally I felt that the process of one-dimensional heat transfer through an insulation is too simple to hide any more subtleties in the last decade of the twentieth century. Indeed, insulations are always treated as layers and shells with uniform thickness [1, 2]. Yet my first try at solving the problem (Section 2) led to a paradoxical conclusion that served as the driving force for the more rigorous study that followed. These developments are reported in the chronological sequence in which they occurred, so that the reader may see how the solution to one problem evolved into the statement for the next problem.

2. PLANE WALL WITH LINEAR TEMPERATURE DISTRIBUTION

In order to see the basic challenge of optimizing the distribution of insulation subject to insulation material constraint, consider the simplest geometry in which this idea can be tried out. Figure 1(a) shows a plane wall of length L and width W (perpendicular to Fig. 1(a)). The wall temperature varies in the longitudinal direction, $T(x)$. We assume for the time being that the wall temperature increases linearly with x, Fig. 1(b)

$$T(x) = T_0 + \frac{x}{L}(T_L - T_0) \quad (1)$$

and that this temperature variation is independent of the amount and distribution of thermal insulation over the length L. This is a good model for the temperature distribution along the outer enclosure of a long counterflow heat exchanger, a stratified water storage tank, or a reheating oven in the steel industry. Linear distribution is used now only for illustration, because it is simple. The general (unspecified) wall temperature distribution will be considered in the next section.

The wall outer surface is separated from the environment of temperature T_0 by a layer of insu-

NOMENCLATURE

b	taper parameter, equation (32)	r	outer radius of pipe wall
C	coefficient, equation (20)	r_o	outer radius of insulation
c_P	specific heat at constant pressure	t	insulation thickness
f	relative insulation volume, equation (23)	t_w	pipe wall thickness
F	integrand of Φ	T	wall temperature, Fig. 1
h	heat transfer coefficient between stream and pipe wall, Fig. 4	T_f	stream temperature, Fig. 4
		T_h	inlet stream temperature
h_o	heat transfer coefficient between insulation and ambient	T_L	wall temperature at $x = L$
		T_{out}	outlet stream temperature
k	thermal conductivity of insulation material	T_0	wall temperature at $x = 0$
		U	overall heat transfer coefficient based on $2\pi r$
k_w	thermal conductivity of pipe wall material	V	insulation volume
K	constant	W	width of plane wall
L	wall length	x	longitudinal coordinate
$\dot{m}$	mass flow rate	Y	dimensionless parameter, equation (22).
n	wall temperature curvature parameter, equation (15)		
N	number of heat transfer units, equation (34)		
q	total heat transfer rate		
q_c	total heat transfer rate through insulation with uniform thickness		
q_{lin}	total heat transfer rate through insulation with linear $t(x)$		

Greek symbols

λ	Lagrange multiplier
Φ	aggregate integral.

Subscripts

avg	average over L
min	minimum
opt	optimal.

lation of thermal conductivity k, and unspecified thickness $t(x)$. The outer surface of the insulation is practically equal to T_0, in other words, the local thermal resistance from the wall surface to the ambient is due entirely to the insulation layer. The wall and its insulation are sufficiently slender in the x direction, so that the heat transfer is oriented in the transversal direction, from $T(x)$ to T_0. The constraint that the amount of insulation material is fixed means that the volume integral

$$V = \int_0^L t(x) W \, dx \qquad (2)$$

has a constant value. An equivalent constraint is that the L-averaged insulation thickness is fixed

$$t_{avg} = \frac{1}{L}\int_0^L t(x)\, dx = \frac{V}{WL}. \qquad (3)$$

The simplest design, of course, is the one in which the insulation is spread evenly over the wall surface $L \times W$, Fig. 1(c)

$$t(x) = t_{avg}, \text{ constant}. \qquad (4)$$

In this case, it is easy to show that the total heat transfer rate from the wall (with linear $T(x)$, equation (1)) to the ambient is

$$q_c = \tfrac{1}{2} k W L \frac{T_L - T_0}{t_{avg}}. \qquad (5)$$

The question next is whether this heat transfer rate (heat 'loss', or heat 'leak' in cryogenics) can be decreased by redistributing the limited amount of insulation more wisely. While looking at the linear $T(x)$ distribution shown in Fig. 1(b), it makes sense to argue that an insulation that is thicker near the $x = L$ end of the wall will be better. Indeed, absolutely no insulation is needed at the other extremity of the wall, because at $x = 0$ the wall–ambient temperature difference is zero.

The simplest way of trying this idea is by having an insulation design in which the thickness $t(x)$ increases linearly from zero at $x = 0$ to a large enough value so that the volume constraint (3) is satisfied, Fig. 1(d)

$$t(x) = 2 t_{avg} \frac{x}{L}. \qquad (6)$$

In this case the total heat transfer rate

$$q = \int_0^L k W \frac{T(x) - T_0}{t(x)} \, dx \qquad (7)$$

is calculated by using the temperature and thickness distributions (1) and (6), and the result is

$$q_{lin} = \tfrac{1}{2} k W L \frac{T_L - T_0}{t_{avg}}. \qquad (8)$$

In this way we reach the paradoxical conclusion that equation (8) is identical to equation (5), i.e. that

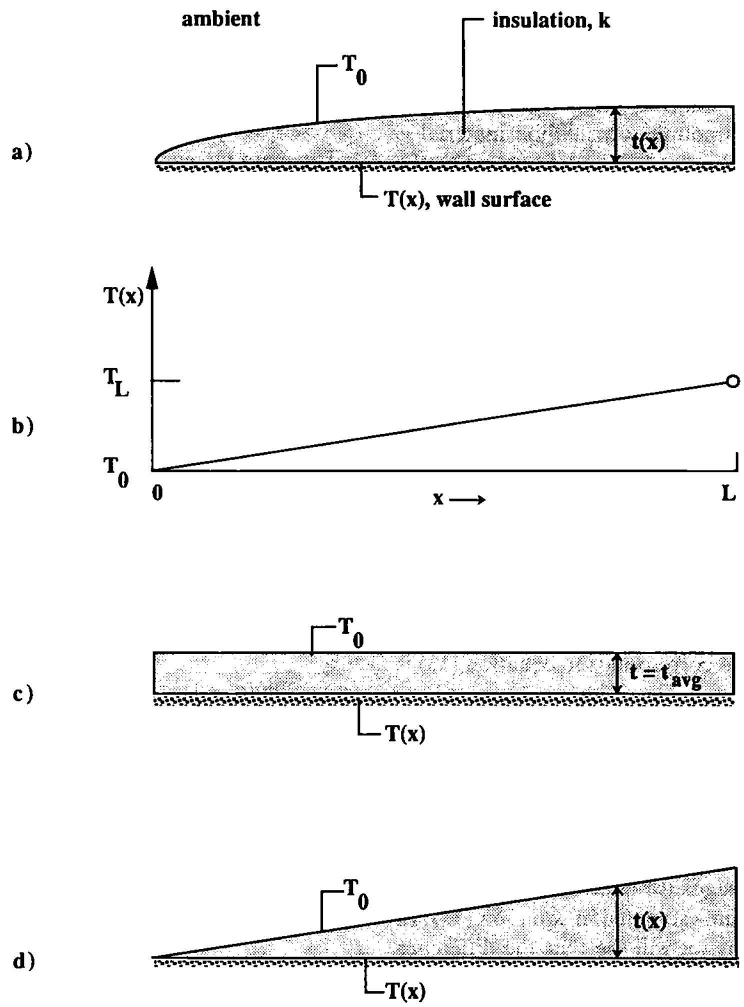

FIG. 1. Plane wall with linear temperature distribution (b) and insulation with various thickness functions (a, c, d).

the tapered insulation layer (6) is as effective as the insulation with uniform thickness (4). It is a conclusion that casts doubt on the method of varying the insulation thickness to minimize heat loss. As we will see in the next section, the variable thickness approach is more challenging (and more subtle) than thought when the linear distribution was chosen (6).

3. OPTIMAL DISTRIBUTION OF INSULATION ON A PLANE WALL

The general problem in which the wall temperature $T(x)$ and the insulation thickness $t(x)$ are not specified consists of minimizing the heat loss integral (7) subject to the volume integral constraint (3). The objective is to find the optimal distribution of insulation, $t_{\text{opt}}(x)$, that minimizes the q integral (7).

The variational-calculus solution is found by noting that the minimization of the integral (7) subject to the integral constraint (3) is analogous to minimizing the aggregate integral [3, 4]

$$\Phi = \int_0^L \left[kW \frac{T(x)-T_0}{t(x)} + \lambda \frac{t(x)}{L} \right] \mathrm{d}x \qquad (9)$$

in which λ is a Lagrange multiplier. Let F be the integrand of Φ, and note that F is a linear combination of the integrands of integrals (7) and (3). The optimal $t(x)$ function that minimizes Φ is the solution to the Euler equation, which in this case is $\partial F/\partial t = 0$. The solution has the form

$$t_{\text{opt}}(x) = K[T(x)-T_0]^{1/2} \qquad (10)$$

in which K is shorthand for the constant $(kWL/\lambda)^{1/2}$. This constant is determined by substituting equation (10) in the volume constraint (3), so that in the end the optimal thickness function is

$$t_{\text{opt}}(x) = \frac{t_{\text{avg}} L}{\displaystyle\int_0^L [T(x)-T_0]^{1/2}\,\mathrm{d}x} [T(x)-T_0]^{1/2}. \qquad (11)$$

The corresponding minimum heat transfer rate through the insulated area $L \times W$ is

$$q_{\min} = \frac{kW}{t_{\rm avg}L}\left\{\int_0^L [T(x)-T_0]^{1/2}\,{\rm d}x\right\}^2. \qquad (12)$$

Worth noting is that the same $t_{\rm opt}(x)$ result is obtained when the amount of insulation material is minimized subject to a fixed rate of heat loss to the ambient. In other words, variational-calculus leads again to equation (11) when the integral (3) is minimized while holding the integral (7) fixed.

In conclusion, for maximum insulation effect the insulation thickness must be proportional to the square root of the local temperature difference across the insulation. In the preceding section, for example, the temperature difference increased linearly in the x direction, and this means that $t_{\rm opt}$ must increase as $x^{1/2}$. Indeed, by substituting equation (1) in the general $t_{\rm opt}(x)$ formula (11) we find

$$t_{\rm opt}(x) = \tfrac{3}{2}t_{\rm avg}\left(\frac{x}{L}\right)^{1/2} \qquad (13)$$

which satisfies also the constraint (3). The minimum heat transfer rate that corresponds to equation (13) is

$$q_{\min} = \tfrac{8}{9}kWL\frac{T_{\rm L}-T_0}{t_{\rm avg}}. \qquad (14)$$

If the earlier designs (4) and (6) are compared with the optimal design (13) for the wall with linearly varying temperature distribution, it is found that the heat loss in the earlier designs (5) and (8) is 12.5% larger than the true minimum estimated in equation (14). In the next two sections we will see that the difference between the constant-t and optimal-t designs can be smaller or larger than this 12.5% difference, depending on the wall shape (plane vs cylindrical) and the wall temperature distribution (linear vs nonlinear). The most important conclusion reached until now is that a finite amount of insulation can be distributed optimally (unevenly, in this case) so that the overall insulation effect is maximized.

4. PLANE WALL WITH NONLINEAR TEMPERATURE DISTRIBUTION

In general, the wall that must be insulated can have a temperature that does not vary linearly with spatial position. For example, in a long reheating oven for the production of laminated steel products, the wall has distinct hot zones according to the positions occupied by the few gas burners. A nonlinear wall temperature $T(x)$ that allows us to investigate the effect of the finite curvature ${\rm d}^2T/{\rm d}x^2$ on the conclusions drawn in the preceding section is the exponential

$$T(x) = T_0+(T_{\rm L}-T_0)\frac{\exp\left(n\dfrac{x}{L}\right)-1}{\exp(n)-1}. \qquad (15)$$

This temperature distribution is illustrated in the inset of Fig. 2. The curvature of the wall temperature function has the same sign as the dimensionless parameter n. This also means that the linear $T(x)$ example of equation (1) represents the $n = 0$ curve of the family represented by equation (15).

The main purpose of Fig. 2 is to show the effect of the wall temperature curvature parameter n on the total rate of heat transfer through the wall surface. The analysis that stands behind the construction of Fig. 2 is left out for the sake of brevity. Plotted on the ordinate is the group $(q_c/q_{\min})-1$, where q_c is the total heat transfer rate when the insulation has uniform thickness ($t = t_{\rm avg}$)

$$q_c = kWL\frac{T_{\rm L}-T_0}{t_{\rm avg}}\cdot\frac{{\rm e}^n-1-n}{n({\rm e}^n-1)}. \qquad (16)$$

The minimum heat transfer rate $q_{\min}$ that corresponds to the optimally distributed insulation material (11) is obtained by substituting equation (15) in equation (12). The result becomes more compact if we present it as the ratio

$$\frac{q_c}{q_{\min}} = \frac{n({\rm e}^n-1-n)}{4\{({\rm e}^n-1)^{1/2}-\tan^{-1}[({\rm e}^n-1)^{1/2}]\}^2} \quad (n>0) \qquad (17a)$$

$$\frac{q_c}{q_{\min}} = \frac{n(n+1-{\rm e}^n)}{\{2(1-{\rm e}^n)^{1/2}-\ln\{[1+(1-{\rm e}^n)^{1/2}]/[1-(1-{\rm e}^n)^{1/2}]\}\}^2} \quad (n<0). \qquad (17b)$$

Figure 2 shows that the reduction in the heat loss through the insulation $[(q_c/q_{\min})-1]$ is greater when the insulation thickness varies optimally and the wall temperature function has positive curvature. When the curvature is negative, the energy savings associated with the optimal distribution of insulation material are of the order of 10%, i.e. of the same order as when the wall temperature varies linearly.

5. CYLINDRICAL WALL WITH LINEAR TEMPERATURE DISTRIBUTION

Consider now the problem of distributing a finite amount of insulation optimally over a cylindrical wall of radius r and length L, Fig. 3. The known temperature distribution of the wall, $T(x)$, is again independent on how the insulation material is distributed. The outer radius of the insulation layer of thickness $t(x)$ is $r+t(x)$, and the insulation temperature at the outer radius is T_0. The insulation volume is fixed

$$V = \int_0^L \pi r^2\left\{\left[1+\frac{t(x)}{r}\right]^2-1\right\}{\rm d}x. \qquad (18)$$

The optimal insulation thickness $t(x)$ can be determined by applying once more the Lagrange multiplier method of Section 3. The total heat transfer rate

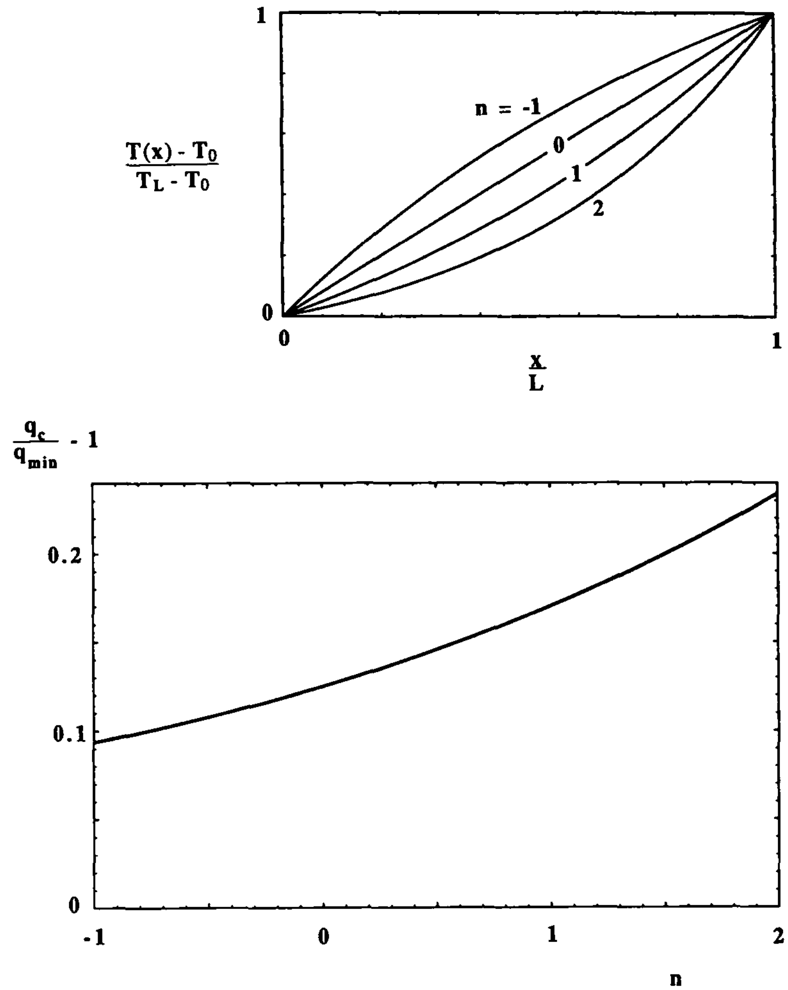

FIG. 2. The effect of a nonlinear wall temperature distribution on the heat loss reduction due to using an insulation with optimal thickness variation.

$$q=\int_0^L \frac{2\pi k[T(x)-T_0]}{\ln[1+t(x)/r]}\,\mathrm{d}x \tag{19}$$

is minimized subject to the integral constraint (18), and the Lagrange multiplier λ is evaluated in the end by invoking the constraint.

In this section we are most interested in how the wall shape affects the plane-wall conclusions reached until now, therefore we omit the analysis and report only on the $t_{opt}(x)$ solution for the particular case when $T(x)$ varies linearly as in equation (1). This $t_{opt}(x)$ solution is given implicitly by

$$\left[1+\frac{t_{opt}(x)}{r}\right]\ln\left[1+\frac{t_{opt}(x)}{r}\right]=C\left(\frac{x}{L}\right)^{1/2} \tag{20}$$

where C is shorthand for the group $[\pi k(T_L-T_0)/\lambda]^{1/2}$. The constant C is evaluated by substituting equation (20) in the integral constraint (18)

$$\frac{2}{C^2}\int_1^Y (y^2-1)y\ln y(\ln y+1)\,\mathrm{d}y=f \tag{21}$$

where

$$Y=1+\frac{t_{opt}(L)}{r} \tag{22}$$

and

$$f=\frac{V}{\pi r^2 L}. \tag{23}$$

Now, if we set $x=L$ in equation (20) we obtain $C=Y\ln Y$, which means that Y is a unique function of C. In conclusion, equation (21) delivers C as a function of the dimensionless parameter f, which is defined as the ratio between the insulation volume and the volume of the cylinder of radius r and length L, equation (23).

The minimum heat transfer rate that corresponds to the insulation with optimal thickness, equation (20), is

$$q_{min}=\frac{4\pi}{C^4}kL(T_L-T_0)\int_0^Y y^3(\ln y)^2(\ln y+1)\,\mathrm{d}y. \tag{24}$$

This is compared in Fig. 3 with the heat transfer

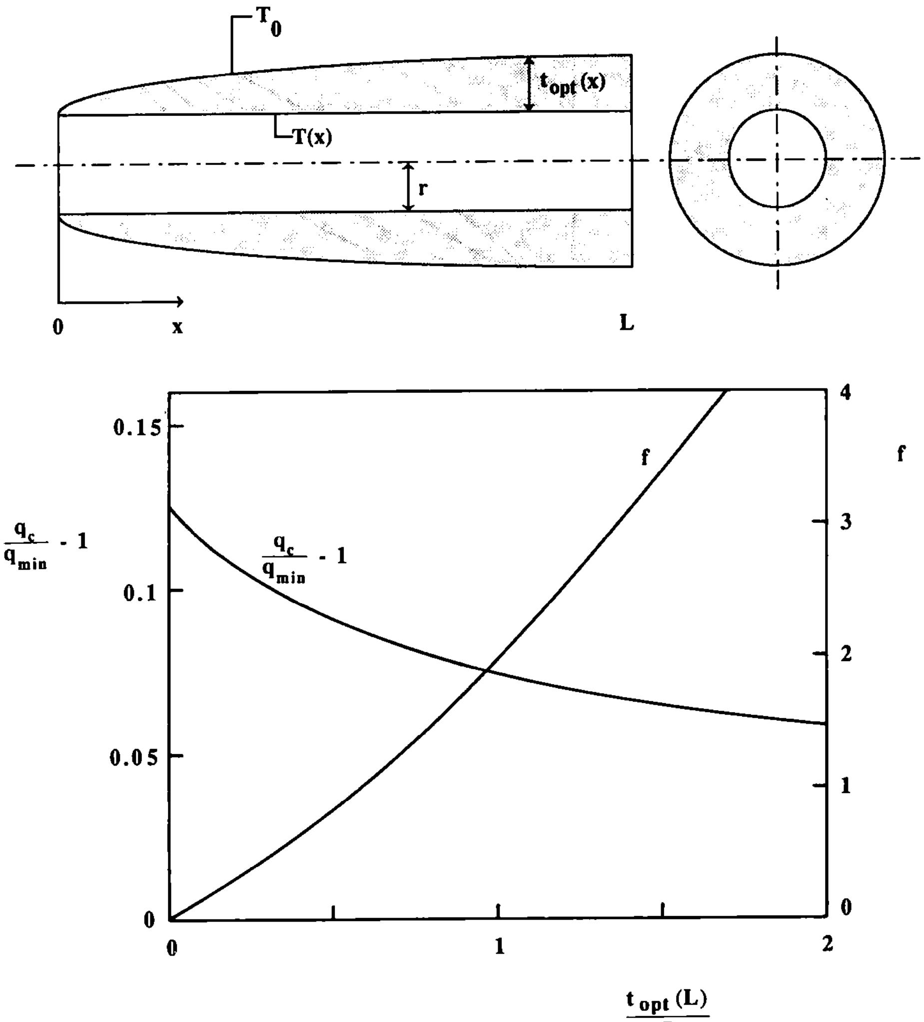

FIG. 3. Cylindrical wall with linear temperature distribution: the heat loss reduction due to using an insulation with optimal thickness variation.

rate that passes through the corresponding constant-thickness insulation ($t = t_{avg}$)

$$q_c = \frac{2\pi k L(T_L - T_0)}{\ln(1+f)}. \quad (25)$$

The volume constraint (18) can be combined with equation (23) to show that $t_{avg}/r = (1+f)^{1/2} - 1$.

Figure 3 shows the relative reduction in heat loss to the ambient, when the design switches from the constant-t configuration to the $t_{opt}(x)$ distribution recommended by equation (20). The parameter that varies freely in the case of the cylindrical geometry is the relative insulation volume f or, on the abscissa, the optimal insulation thickness at the $x = L$ end, $t_{opt}(L)/r$.

The limit $f \to 0$ represents an insulation that is so thin (relative to r) that it can be treated as an insulation mounted on a plane wall. This means that the solution developed in Section 3 for the plane wall with linear $T(x)$ is equivalent to setting $f = 0$ in Fig. 3. As f (or $t_{opt}(L)/r$) increases, the insulation looks more like a thick shell around the cylinder, and the ratio q_c/q_{min} decreases. In conclusion, the heat loss reduction due to using an optimal distribution of insulation on a cylinder is smaller than the reduction registered if the wall is plane.

6. STREAM SUSPENDED IN AN ENVIRONMENT OF DIFFERENT TEMPERATURE

In all the variants of the optimization problem discussed until now it was assumed that the wall temperature distribution $T(x)$ is not affected by the amount of insulation and the manner in which this amount is distributed over the wall length. In this section, this modelling feature is discarded and attention is turned to Fig. 4.

One of the simplest and most basic configurations in which the wall temperature distribution is intimately coupled to the insulation performance is the stream suspended in an environment of different temperature.

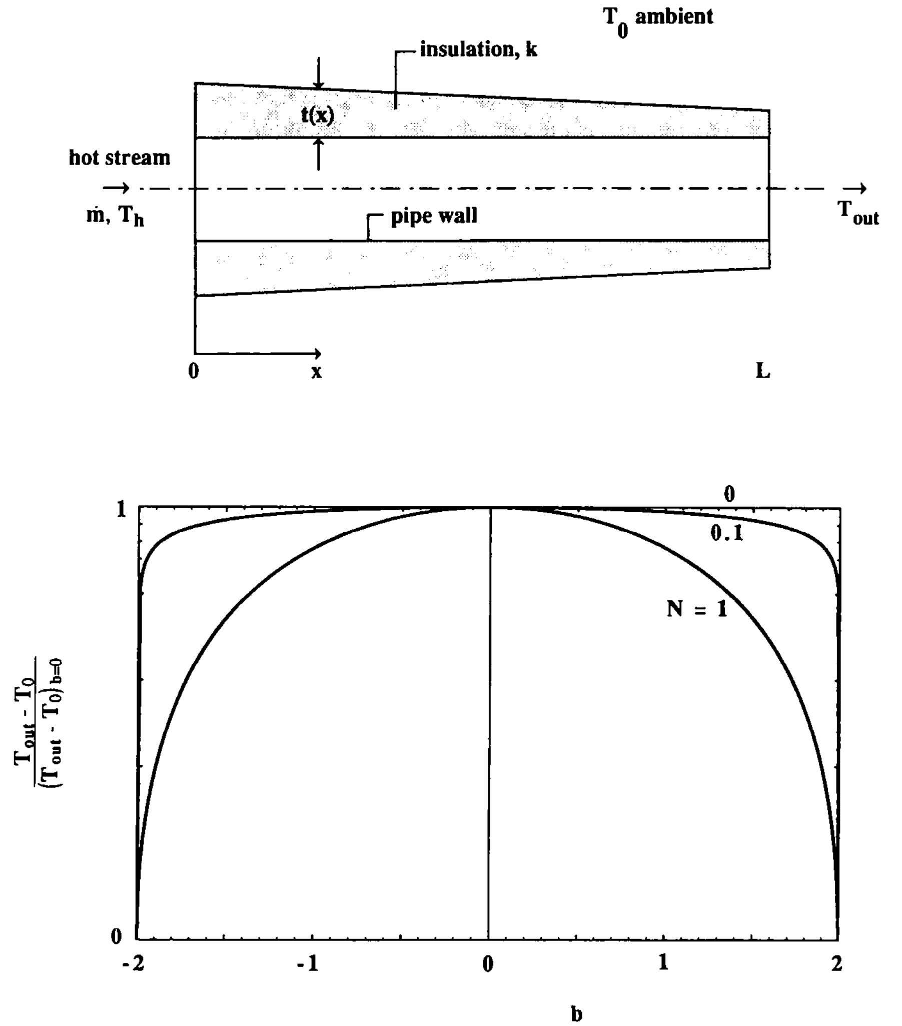

FIG. 4. Insulated stream suspended in an environment of different temperature: the effect of tapering the layer of insulation.

This is one of the most common features of thermal design, from power plants and chemical process plants, to piping in large buildings. The bulk temperature of the ducted (single-phase) stream T_f decreases longitudinally only because of the loss of heat through the insulation. The function of the insulated duct is to deliver the stream with an outlet temperature (T_{out}) that resembles as closely as possible the high inlet temperature (T_h).

The duct can have any cross-sectional shape, however, for better illustration a pipe with the outer radius r is assumed. The outer radius of the insulation layer is $r_o(x) = r + t(x)$, and the thickness $t(x)$ is not necessarily small relative to r. The assumptions that the insulation accounts for the entire thermal resistance between wall and ambient are also dropped. In Fig. 4, the overall heat transfer coefficient U between the local bulk temperature of the stream $T_f(x)$ and the environment T_0 is given by (e.g. see p. 85 of Incropera and DeWitt [1])

$$\frac{1}{U2\pi r} = \frac{1}{h_o 2\pi r_o} + \frac{\ln\,[1 + t(x)/r]}{2\pi k} + \frac{t_w}{k_w 2\pi r} + \frac{1}{h2\pi r}. \tag{26}$$

The four resistances on the right-hand side represent, in order, convection outside the insulation (constant h_o), conduction through the insulation, conduction through the pipe wall (thickness $t_w \ll r$, thermal conductivity k_w), and fully developed convection inside the pipe (constant h). Note that U is a function of x, because of $t(x)$.

The first-law statement for a control volume of length $\mathrm{d}x$ and radius r is

$$-\dot{m}c_P\,\mathrm{d}T_f = U2\pi r(T_f - T_0)\,\mathrm{d}x \tag{27}$$

in which $\dot{m}c_P$ is the capacity rate of the stream. Next, we integrate equation (27) from $x = 0$ (where $T_f = T_h$) to L (where $T_f = T_{out}$), and obtain the integral

$$\ln\frac{T_h - T_0}{T_{out} - T_0} = \int_0^L \frac{U2\pi r}{\dot{m}c_P}\,\mathrm{d}x. \tag{28}$$

This integral must be minimized, because the purpose of the insulation is to maintain the highest possible T_{out}. The maximization of the integral (28) is subject to the volume constraint (18). The variational-calculus problem boils down to minimizing the aggregate integral

$$\Phi = \int_0^L \left\{ \frac{U 2\pi r}{\dot{m} c_P} + \lambda \pi r^2 \left[\left(1 + \frac{t}{r}\right)^2 - 1 \right] \right\} \mathrm{d}x \quad (29)$$

whose integrand is labelled F, in other words, $\{\ \} = F$. The optimal function $t(x)$ is the solution to the Euler equation

$$\frac{\partial F}{\partial t} = 0. \quad (30)$$

This equation can be written down by using equation (26) for $U[t(x)]$. This last analytical step is not necessary if we notice that U decreases when t increases (assuming that r is greater than the critical radius of insulation), while the second term in F (the one multiplied by λ) increases when t increases. This means, first, that F has a minimum with respect to t. That minimum can be pinpointed by solving equation (30), but since all the other quantities that will be present in that equation are x independent, the $t(x)$ solution of equation (30) is simply

$$t_{\text{opt}} = \text{constant}. \quad (31a)$$

The actual constant is evaluated by forcing t_{opt} to obey the volume constraint (18)

$$t_{\text{opt}} = r \left[\left(\frac{V}{\pi r^2 L} + 1 \right)^{1/2} - 1 \right]. \quad (31b)$$

The same conclusion is reached if the roles of the two integrals in Φ are reversed, i.e. if the amount of insulation is minimized subject to a fixed rate of heat transfer to the ambient. It is a general conclusion, in view of the many features included in the heat transfer model (26). This conclusion differs from what we found in Sections 2–5, because in those earlier examples we considered wall temperature distributions that do not depend on the insulation that is applied on them.

We learned in this section that the best insulation is also the simplest, i.e. the one with uniform thickness. It is fascinating that what engineers have been doing all along (no doubt, for ease of installation and expediency) is actually the optimal way of using a *limited amount* of insulation material.

Without knowing this general conclusion, it would have been reasonable to argue that the better insulation must be thicker near the inlet (hot end), because in that region the stream–ambient temperature difference is larger than downstream. One option is to use the tapered insulation shown in the upper part of Fig. 4, which is represented by

$$t(x) = t_{\text{avg}} \left[1 - b \left(\frac{x}{L} - \frac{1}{2} \right) \right] \quad (32)$$

if it is assumed for simplicity that $t_{\text{avg}} \ll r$. The dimensionless parameter b accounts for the taper, and has the range $-2 < b < 2$. If we further assume (as in Sections 2–5) that the stream–ambient thermal resistance is due entirely to the layer of insulation, in place of equation (28) we obtain

$$\frac{T_{\text{out}} - T_0}{T_{\text{h}} - T_0} = \exp\left(-\frac{N}{b} \ln \frac{1 + b/2}{1 - b/2} \right). \quad (33)$$

Parameter N is the 'number of heat transfer units'

$$N = \frac{(k/t_{\text{avg}}) 2\pi r L}{\dot{m} c_P}. \quad (34)$$

In the special case when the insulation thickness is uniform ($b = 0$), equation (33) reduces to

$$\frac{(T_{\text{out}} - T_0)_{b=0}}{T_{\text{h}} - T_0} = \exp(-N). \quad (35)$$

The tapered and constant-thickness designs can be compared by examining the ratio $(T_{\text{out}} - T_0)/(T_{\text{out}} - T_0)_{b=0}$. This ratio is plotted vs b and N in the lower part of Fig. 4. It reconfirms the conclusion (31) that the best design (highest T_{out}) is the one with uniform distribution of insulation ($b = 0$). The uniform-t design is superior especially when the insulation supply (t_{avg}) is so small that the order of N exceeds 0.1.

7. CONCLUSIONS

The main conclusions of this study are that:

(1) it is possible to distribute a finite amount of insulation in a certain, nontrivial way that minimizes the total heat transfer rate from a nonisothermal wall to the ambient;

(2) when the wall is plane, the optimal thickness of the insulation varies as the square root of the local wall–ambient temperature difference;

(3) the heat loss reduction due to using an insulation with optimal thickness becomes greater as the curvature of the wall temperature distribution ($\mathrm{d}^2 T/\mathrm{d}x^2$) increases;

(4) the heat loss reduction due to using an insulation with optimal thickness on a cylindrical wall is smaller than on the corresponding plane wall; and

(5) the best insulation for a single-phase stream suspended in an environment of different temperature is the insulation with uniform thickness.

REFERENCES

1. F. P. Incropera and D. P. DeWitt, *Fundamentals of Heat and Mass Transfer*, 3rd Edn, pp. 79–104. Wiley, New York (1990).
2. W. M. Rohsenow, J. P. Hartnett and E. N. Ganic (eds), *Handbook of Heat Transfer Applications*, 2nd Edn, pp. (8-8)–(8-11). McGraw-Hill, New York (1985).
3. F. B. Hildebrand, *Advanced Calculus for Applications*, p. 355. Prentice-Hall, Englewood Cliffs, New Jersey (1962).
4. A. Bejan, *Advanced Engineering Thermodynamics*, p. 722. Wiley, New York (1988).